Optical solitons in birefringent fibers with parabolic law nonlinearity

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This paper studies the propagation of optical solitons through birefringent fibers with parabolic law nonlinearity. The Hamiltonian perturbations that are inter-modal dispersion, self-steepening, third-order dispersion and nonlinear dispersions are taken into account. Both, Riccati equation expansion method and Jacobian elliptic equation expansion method are used. Finally, analytical solutions that are Jacobian elliptic periodic traveling wave solutions, periodic solutions, unbounded solutions, singular solutions, bright and dark soliton solutions are obtained under several constraint conditions.

Keywords: solitons, parabolic law nonlinearity, birefringent fibers, Jacobian elliptic equation, Riccati equation.

1. Introduction

Optical solitons, the most ideal carriers of information, have important application features in the optical communications and ultra-fast signal processing systems [1–5]. Most of the existing papers mainly focus on the optical solitons in the polarization preserving fibers, while there are very few papers that study the optical solitons in the birefringent fibers [6–14]. So the key idea of this paper is to seek exact soliton solutions to the birefringent fibers with Hamiltonian perturbations and parabolic law nonlinearity.

Birefringence is a natural phenomenon that occurs in optical fibers [6, 8]. The optical pulse will split into two orthogonally polarized pulses that have different propagation constants and group velocities, because it is very difficult to have delicate circularly symmetry for optical fibers [8].
In the presence of strong Hamiltonian type perturbations, the governing equation for the propagation of optical solitons through birefringent fibers with parabolic law nonlinearity is given by the following Hirota equations:

\[ i q_t + a_1 q_{xx} + \left( b_1 |q|^2 + c_1 |r|^2 \right) q + \left( d_1 |q|^4 + e_1 |q|^2 |r|^2 + f_1 |r|^4 \right) q + \right. \\
+ i \lambda_1 q_x + i s_1 |q|^2 q + i \mu_1 |q|^2 q + i \theta_1 |q|^2 q_x + i \gamma_1 q_{xxx} = 0 \]  

\[ i r_t + a_2 r_{xx} + \left( b_2 |r|^2 + c_2 |q|^2 \right) r + \left( d_2 |r|^4 + e_2 |r|^2 |q|^2 + f_2 |q|^4 \right) r + \\
\left. + i \lambda_2 r_x + i s_2 |r|^2 r + i \mu_2 |r|^2 r + i \theta_2 |r|^2 r_x + i \gamma_2 r_{xxx} = 0 \]  

In Equations (1) and (2), the unknown functions \( q(x, t) \) and \( r(x, t) \) are the optical wave profiles for the two components in birefringent fibers; \( x \) and \( t \) represent the spatial and temporal variables, respectively.

For \( l = 1, 2 \), the constant parameters \( a_l, b_l, c_l, \lambda_l, s_l \) and \( \gamma_l \) are, respectively, the parameters of the group velocity dispersion (GVD), self-phase modulation (SPM), cross-phase modulation (XPM), inter-modal dispersion (IMD), self-steepening and third-order dispersion (TOD) for the two polarized pulses. The terms with \( d_l, e_l, \) and \( f_l \) are associated with the quintic terms of the parabolic (cubic-quintic) law nonlinearity \([7, 8]\). Finally, \( \mu_l \) and \( \theta_l \) are the nonlinear dispersions.

The aim of the present work is to construct the Jacobian elliptic periodic traveling wave solutions, periodic solutions, unbounded solutions, singular solutions, singular, bright and dark soliton solutions in the birefringent fibers with Hamiltonian perturbations and parabolic law nonlinearity. The strong Hamiltonian type perturbations that are IMD, self-steepening, TOD and nonlinear dispersions are taken into consideration. The integration methods are the Riccati equation expansion method and Jacobian elliptic equation expansion method. Several constraint conditions for analytical solutions to exist are displayed.

In order to obtain exact solutions to Eqs. (1) and (2), making the hypothesis in the form \([6–9]\):

\[ q(x, t) = A_1 P_1[\eta(x, t)] \exp \left[ i \phi_1(x, t) \right] \]  
\[ r(x, t) = A_2 P_2[\eta(x, t)] \exp \left[ i \phi_2(x, t) \right] \]  

where \( \eta = B(x - vt) \) and \( \phi_l = -\kappa_l x + \omega_l t + \theta_l ; P_l(\eta) \) and \( \phi_l(x, t) \) for \( l = 1, 2 \) are the amplitude and phase components of the two solitons, respectively; \( A_l, B \) and \( v \) represent the amplitude, width and velocity of the solitons. Additionally, \( \kappa_l \) are frequencies of the two solitons, \( \omega_l \) are the wave numbers, while \( \theta_l \) are the phase constants.
Substituting (3) and (4) into (1) and (2), and separating the real and imaginary parts, respectively, one obtains

\[
\begin{align*}
-\left(\omega_l - \lambda_i \kappa_l + a_i \kappa_l^2 + \gamma_l \kappa_l^3\right)P_l + c_i A_i^2 P_l P_l' + d_i A_i^4 P_l^5 + e_i A_i^2 A_i^2 P_l^3 P_l'^2 + \\
+f_i A_i^4 P_l P_l' + (b_i + s_i \kappa_l + \theta_i \kappa_l) A_i^2 P_l^3 + (a_i + 3 \gamma_l \kappa_l) B^2 P_l' = 0 \\
\left(\lambda_l - 2a_i \kappa_l - 3 \gamma_i \kappa_l^2 - \nu\right) P_l' + (3s_i + 2\mu_i + \theta_i) A_i^2 P_l^2 P_l' + \gamma_l B^2 P_l'' = 0
\end{align*}
\]

(5)  (6)

for \(l = 1, 2\) and \(\bar{\ell} = 3 - l\).

2. Riccati equation expansion method

Assume that \(P_l(\eta)\) satisfies

\[
P_l'(\eta) = a + bP_l^2(\eta)
\]

(7)

where \(a\) and \(b\) are the nonzero real constants. Equation (7) is the famous Riccati equation [15–17], the solutions of which are listed in Table 1.

<table>
<thead>
<tr>
<th>(ab &gt; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_l(\eta) = \frac{\sqrt{ab}}{b} \tan(\sqrt{ab} \ \eta))</td>
</tr>
<tr>
<td>(P_l(\eta) = -\frac{\sqrt{ab}}{b} \cot(\sqrt{ab} \ \eta))</td>
</tr>
</tbody>
</table>

<table>
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<tr>
<th>(ab &lt; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P_l(\eta) = -\frac{\sqrt{-ab}}{b} \tanh(\sqrt{-ab} \ \eta))</td>
</tr>
<tr>
<td>(P_l(\eta) = -\frac{\sqrt{-ab}}{b} \coth(\sqrt{-ab} \ \eta))</td>
</tr>
</tbody>
</table>

Substituting the assumption (7) into Eqs. (5) and (6) yields

\[
\begin{align*}
-\left(\omega_l - \lambda_i \kappa_l + a_i \kappa_l^2 + \gamma_l \kappa_l^3\right)P_l + c_i A_i^2 P_l P_l' + d_i A_i^4 P_l^5 + e_i A_i^2 A_i^2 P_l^3 P_l'^2 + \\
+f_i A_i^4 P_l P_l' + (b_i + s_i \kappa_l + \theta_i \kappa_l) A_i^2 P_l^3 + (a_i + 3 \gamma_l \kappa_l) B^2 \left(2abP_l + 2b^2 P_l^3\right) = 0
\end{align*}
\]

(8)

\[
\begin{align*}
\left(\lambda_l - 2a_i \kappa_l - 3 \gamma_i \kappa_l^2 - \nu\right) \left(a + bP_l^2\right) + (3s_i + 2\mu_i + \theta_i) A_i^2 P_l^2 \left(a + bP_l^2\right) + \\
+ \gamma_l B^2 \left(2a^2 b + 8ab^2 P_l^2 + 6b^3 P_l^4\right) = 0
\end{align*}
\]

(9)
Then using the homogeneous balance principle, from Eqs. (8) and (9), setting the coefficients of each power of $P_l(\eta)$ to zero gives:

\begin{align}
\omega_l &= \lambda_l \kappa_l - a_l \kappa_l^2 - \gamma_l \kappa_l^3 + 2ab(a_l + 3 \gamma_l \kappa_l)B^2 \\
(b_l + s_l \kappa_l + \theta_l \kappa_l) A_l^2 + 2b^2(a_l + 3 \gamma_l \kappa_l) B^2 + c_l A_l^2 &= 0 \\
d_l A_l^4 + e_l A_l^2 A_f^2 + f_l A_f^4 &= 0 \\
\nu &= \lambda_l - 2a_l \kappa_l - 3 \gamma_l \kappa_l^2 + 2ab \gamma_l B^2 \\
\nu &= \lambda_l - 2a_l \kappa_l - 3 \gamma_l \kappa_l^2 + 8ab \gamma_l B^2 + \frac{a(3s_l + 2\mu_l + \theta_l)}{b} A_l^2 \\
(3s_l + 2\mu_l + \theta_l) A_l^2 + 6b^2 \gamma_l B^2 &= 0
\end{align}

It needs to be noted that upon equating the two values of the solitons velocities from (13) and (14) also yields the same relation as given by (15).

Equating the two values of the soliton velocity $\nu$, for $l = 1, 2$, from Eq. (13) gives the width of the soliton as

\begin{equation}
B = \left[ \frac{(\lambda_l - \lambda_T) - 2(a_l \kappa_l - a_T \kappa_T) - 3(\gamma_l \kappa_l^2 - \gamma_T \kappa_T^2)}{2ab(\gamma_l - \gamma_T)} \right]^{1/2}
\end{equation}

which introduces the constraint condition

\begin{equation}
ab(\gamma_l - \gamma_T) \left[ (\lambda_l - \lambda_T) - 2(a_l \kappa_l - a_T \kappa_T) - 3(\gamma_l \kappa_l^2 - \gamma_T \kappa_T^2) \right] > 0
\end{equation}

From Eq. (15), the amplitude of the solitons are given by

\begin{equation}
A_l = \left\{ \frac{3b \gamma_l \left[ (\lambda_l - \lambda_T) - 2(a_l \kappa_l - a_T \kappa_T) - 3(\gamma_l \kappa_l^2 - \gamma_T \kappa_T^2) \right]}{a(3s_l + 2\mu_l + \theta_l)(\gamma_l - \gamma_T)} \right\}^{1/2}
\end{equation}

with the constraint condition

\begin{equation}
\gamma_l (3s_l + 2\mu_l + \theta_l) < 0
\end{equation}

Additionally, Equations (11) and (12) pose other two constraint conditions that are given by
Hence, finally the singular solutions, dark and singular soliton solutions for the birefringent fibers with parabolic law nonlinearity are obtained, which are listed as follows.

**Case 1** – when $ab > 0$, Eqs. (1) and (2) admit the singular periodic solutions that are given by

$$q(x, t) = \frac{\sqrt{ab}}{b} A_1 \tan \left[ \sqrt{ab} B(x - vt) \right] \exp \left[ i(-\kappa_1 x + \omega_1 t + \theta_1) \right]$$

(22)

$$r(x, t) = \frac{\sqrt{ab}}{b} A_2 \tan \left[ \sqrt{ab} B(x - vt) \right] \exp \left[ i(-\kappa_2 x + \omega_2 t + \theta_2) \right]$$

(23)

$$q(x, t) = -\frac{\sqrt{ab}}{b} A_1 \cot \left[ \sqrt{ab} B(x - vt) \right] \exp \left[ i(-\kappa_1 x + \omega_1 t + \theta_1) \right]$$

(24)

$$r(x, t) = -\frac{\sqrt{ab}}{b} A_2 \cot \left[ \sqrt{ab} B(x - vt) \right] \exp \left[ i(-\kappa_2 x + \omega_2 t + \theta_2) \right]$$

(25)

**Case 2** – when $ab < 0$, Eqs. (1) and (2) admit the dark soliton solutions that are given by

$$q(x, t) = -\frac{\sqrt{-ab}}{b} A_1 \tanh \left[ \sqrt{-ab} B(x - vt) \right] \exp \left[ i(-\kappa_1 x + \omega_1 t + \theta_1) \right]$$

(26)

$$r(x, t) = -\frac{\sqrt{-ab}}{b} A_2 \tanh \left[ \sqrt{-ab} B(x - vt) \right] \exp \left[ i(-\kappa_2 x + \omega_2 t + \theta_2) \right]$$

(27)

and the singular soliton solutions that are given by

$$q(x, t) = -\frac{\sqrt{-ab}}{b} A_1 \coth \left[ \sqrt{-ab} B(x - vt) \right] \exp \left[ i(-\kappa_1 x + \omega_1 t + \theta_1) \right]$$

(28)

$$r(x, t) = -\frac{\sqrt{-ab}}{b} A_2 \coth \left[ \sqrt{-ab} B(x - vt) \right] \exp \left[ i(-\kappa_2 x + \omega_2 t + \theta_2) \right]$$

(29)

where the amplitude and width of the solitons are given by Eqs. (18) and (16) respectively, while the velocity of the solitons are given by Eq. (13) or (14) and finally
the wave numbers are given by Eq. (10). The constraint conditions for analytical solutions to exist are given by Eqs. (17) and (19)–(21).

3. Jacobian elliptic equation expansion method

Assume that \( P_l(\eta) \) satisfies

\[
P_l^2(\eta) = g_0 + g_2 P_l^2(\eta) + g_4 P_l^4(\eta)
\]

where \( g_0, g_2 \) and \( g_4 \) are the nonzero real constants. Eq. (30) is Jacobian elliptic equation, the solutions of which are listed in [2, 18–20].

Substituting the assumption (30) into Eqs. (5) and (6) yields

\[
-\left(\omega_l - \lambda_l \kappa_l + a_l \kappa_l^2 + \gamma_l \kappa_l^3\right) P_l + c_l A_l^2 P_l^2 + d_l A_l^4 P_l^5 + e_l A_l^2 A_l^2 P_l^3 P_l^1 +

+ f_l A_l^4 P_l^3 P_l^1 + (b_l + s_l \kappa_l + \theta_l \kappa_l) A_l^2 P_l^3 + (a_l + 3 \gamma_l \kappa_l) B^2 \left(g_2 P_l + 2 g_4 P_l^3\right) = 0
\]

(31)

\[
\left(\lambda_l - 2 a_l \kappa_l - 3 \gamma_l \kappa_l^2 - \nu\right)^2 + 2 \left(\lambda_l - 2 a_l \kappa_l - 3 \gamma_l \kappa_l^2 - \nu\right) (3 s_l + 2 \mu_l + \theta_l) A_l^2 P_l^2 +

+ (3 s_l + 2 \mu_l + \theta_l) A_l^4 P_l^4 = \gamma_l^2 B^4 \left(g_2 + 6 g_4 P_l^2\right)^2
\]

(32)

Then using the homogeneous balance principle, from Eqs. (31) and (32), setting the coefficients of each power of \( P_l(\eta) \) to zero gives

\[
\omega_l = g_2 (a_l + 3 \gamma_l \kappa_l) B^2 + \lambda_l \kappa_l - a_l \kappa_l^2 - \gamma_l \kappa_l^3
\]

(33)

\[
(b_l + s_l \kappa_l + \theta_l \kappa_l) A_l^2 + c_l A_l^2 + 2 g_4 (a_l + 3 \gamma_l \kappa_l) B^2 = 0
\]

(34)

\[
d_l A_l^4 + e_l A_l^2 A_l^2 + f_l A_l^4 = 0
\]

(35)

\[
v = \lambda_l - 2 a_l \kappa_l - 3 \gamma_l \kappa_l^2 + g_2 \gamma_l B^2
\]

(36)

\[
v = \lambda_l - 2 a_l \kappa_l - 3 \gamma_l \kappa_l^2 - \frac{6 g_2 g_4 \gamma_l^2 B^4}{(3 s_l + 2 \mu_l + \theta_l) A_l^2}
\]

(37)

\[
(3 s_l + 2 \mu_l + \theta_l) A_l^2 = 6 g_4 \gamma_l B^2
\]

(38)

It needs to be noted that equating the two values of the solitons velocities from (36) and (37) also yields the same relation as given by (38).
Table 2. Jacobian elliptic periodic traveling wave solutions to Eqs. (1) and (2).

<table>
<thead>
<tr>
<th>$g_0$</th>
<th>$g_2$</th>
<th>$g_4$</th>
<th>$q(x, t)$</th>
<th>$r(x, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^2 E^2$</td>
<td>$-\lambda^2 (1 + m^2)$</td>
<td>$\lambda^2 m^2 / E^2$</td>
<td>$EA_1 sn [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 sn [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$\lambda^2 E^2 (1 - m^2)$</td>
<td>$\lambda^2 (2m^2 - 1)$</td>
<td>$-\lambda^2 m^2 / E^2$</td>
<td>$EA_1 cn [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 cn [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$-\lambda^2 E^2 (1 - m^2)$</td>
<td>$\lambda^2 (2m^2 - 1)$</td>
<td>$-\lambda^2 / E^2$</td>
<td>$EA_1 dn [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 dn [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$\lambda^2 m^2 E^2$</td>
<td>$-\lambda^2 (1 + m^2)$</td>
<td>$\lambda^2 / E^2$</td>
<td>$EA_1 nc [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 nc [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$-\lambda^2 m^2 E^2$</td>
<td>$\lambda^2 (2m^2 - 1)$</td>
<td>$-\lambda^2 (1 - m^2) / E^2$</td>
<td>$EA_1 nd [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 nd [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$-\lambda^2 E^2$</td>
<td>$\lambda^2 (2m^2 - 1)$</td>
<td>$-\lambda^2 (1 - m^2) / E^2$</td>
<td>$EA_1 sc [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 sc [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$\lambda^2 E^2$</td>
<td>$\lambda^2 (2m^2 - 1)$</td>
<td>$\lambda^2 / E^2$</td>
<td>$EA_1 sd [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 sd [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$\lambda^2 E^2 (1 - m^2)$</td>
<td>$\lambda^2 (2m^2 - 1)$</td>
<td>$\lambda^2 / E^2$</td>
<td>$EA_1 cs [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 cs [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
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<tr>
<td>$\lambda^2 E^2$</td>
<td>$-\lambda^2 (1 + m^2)$</td>
<td>$\lambda^2 m^2 / E^2$</td>
<td>$EA_1 cd [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 cd [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$-\lambda^2 m^2 (1 - m^2) E^2$</td>
<td>$\lambda^2 (2m^2 - 1)$</td>
<td>$\lambda^2 / E^2$</td>
<td>$EA_1 ds [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 ds [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
</tbody>
</table>

Table 3. Trigonometric periodic solutions to Eqs. (1) and (2).

<table>
<thead>
<tr>
<th>$g_0$</th>
<th>$g_2$</th>
<th>$g_4$</th>
<th>$q(x, t)$</th>
<th>$r(x, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^2 E^2$</td>
<td>$-\lambda^2$</td>
<td>0</td>
<td>$EA_1 sin [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 sin [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$\lambda^2 E^2$</td>
<td>$-\lambda^2$</td>
<td>0</td>
<td>$EA_1 cos [\lambda B(x - vt)] exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 cos [\lambda B(x - vt)] exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
</tbody>
</table>
Table 4. Unbounded solutions to Eqs. (1) and (2).

<table>
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<tr>
<th>$g_0$</th>
<th>$g_2$</th>
<th>$g_4$</th>
<th>$q(x, t)$</th>
<th>$r(x, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-\lambda^2 E^2$</td>
<td>$\lambda^2$</td>
<td>0</td>
<td>$EA_1 \cosh(\lambda B(x - vt)) \exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 \cosh(\lambda B(x - vt)) \exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$\lambda^2 E^2$</td>
<td>$\lambda^2$</td>
<td>0</td>
<td>$EA_1 \sinh(\lambda B(x - vt)) \exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 \sinh(\lambda B(x - vt)) \exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
</tbody>
</table>

Table 5. Singular periodic solutions to Eqs. (1) and (2).

<table>
<thead>
<tr>
<th>$g_0$</th>
<th>$g_2$</th>
<th>$g_4$</th>
<th>$q(x, t)$</th>
<th>$r(x, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$-\lambda^2$</td>
<td>$\lambda^2/E^2$</td>
<td>$EA_1 \csc(\lambda B(x - vt)) \exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 \csc(\lambda B(x - vt)) \exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>0</td>
<td>$-\lambda^2$</td>
<td>$\lambda^2/E^2$</td>
<td>$EA_1 \sec(\lambda B(x - vt)) \exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 \sec(\lambda B(x - vt)) \exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$\lambda^2 E^2$</td>
<td>$2\lambda^2$</td>
<td>$\lambda^2/E^2$</td>
<td>$EA_1 \tan(\lambda B(x - vt)) \exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 \tan(\lambda B(x - vt)) \exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$\lambda^2 E^2$</td>
<td>$2\lambda^2$</td>
<td>$\lambda^2/E^2$</td>
<td>$EA_1 \cot(\lambda B(x - vt)) \exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 \cot(\lambda B(x - vt)) \exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
</tbody>
</table>

Table 6. Singular, dark and bright soliton solutions to Eqs. (1) and (2).

<table>
<thead>
<tr>
<th>$g_0$</th>
<th>$g_2$</th>
<th>$g_4$</th>
<th>$q(x, t)$</th>
<th>$r(x, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda^2 E^2$</td>
<td>$-2\lambda^2$</td>
<td>$\lambda^2/E^2$</td>
<td>$EA_1 \coth(\lambda B(x - vt)) \exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 \coth(\lambda B(x - vt)) \exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>0</td>
<td>$\lambda^2$</td>
<td>$\lambda^2/E^2$</td>
<td>$EA_1 \csch(\lambda B(x - vt)) \exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 \csch(\lambda B(x - vt)) \exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>$\lambda^2 E^2$</td>
<td>$-2\lambda^2$</td>
<td>$\lambda^2/E^2$</td>
<td>$EA_1 \tanh(\lambda B(x - vt)) \exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 \tanh(\lambda B(x - vt)) \exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
<tr>
<td>0</td>
<td>$\lambda^2$</td>
<td>$-\lambda^2/E^2$</td>
<td>$EA_1 \sech(\lambda B(x - vt)) \exp[i (-\kappa_1 x + \omega_1 t + \theta_1)]$</td>
<td>$EA_2 \sech(\lambda B(x - vt)) \exp[i (-\kappa_2 x + \omega_2 t + \theta_2)]$</td>
</tr>
</tbody>
</table>
Equating the two values of the soliton velocity \( v \), for \( l = 1, 2 \), from Eq. (36) gives the width of the soliton as

\[
B = \left[ \frac{(\lambda_l - \lambda_T) - 2(a_l \kappa_l - a_T \kappa_T) - 3(\gamma_l \kappa_l^2 - \gamma_T \kappa_T^2)}{g_2(\gamma_l - \gamma_T)} \right]^{1/2}
\]

(39)

which poses the constraint condition

\[
g_2(\gamma_l - \gamma_T) \left[ (\lambda_l - \lambda_T) - 2(a_l \kappa_l - a_T \kappa_T) - 3(\gamma_l \kappa_l^2 - \gamma_T \kappa_T^2) \right] > 0
\]

(40)

From Eq. (38), the amplitudes of the solitons are given by

\[
A_l = \left\{ \frac{6g_4 \gamma_l \left[ (\lambda_l - \lambda_T) - 2(a_l \kappa_l - a_T \kappa_T) - 3(\gamma_l \kappa_l^2 - \gamma_T \kappa_T^2) \right]}{g_2(3s_l + 2\mu_l + \theta_l)(\gamma_l - \gamma_T)} \right\}^{1/2}
\]

(41)

with the constraint condition

\[
g_4 \gamma_l \left( 3s_l + 2\mu_l + \theta_l \right) > 0
\]

(42)

Additionally, Equations (34) and (35) pose other two constraint conditions that are given by

\[
\frac{3 \gamma_l (b_l + s_l \kappa_l + \theta_l \kappa_l)}{3s_l + 2\mu_l + \theta_l} + \frac{3c_l \gamma_T}{3s_T + 2\mu_T + \theta_T} + (a_l + 3\gamma_l \kappa_l) = 0
\]

(43)

\[
\frac{d_l \gamma_l^2}{(3s_l + 2\mu_l + \theta_l)^2} + \frac{e_l \gamma_l \gamma_T}{(3s_l + 2\mu_l + \theta_l)(3s_T + 2\mu_T + \theta_T)} + \frac{f_l \gamma_T^2}{(3s_T + 2\mu_T + \theta_T)^2} = 0
\]

(44)

Hence, finally the explicit Jacobian elliptic periodic traveling wave solutions for the birefringent fibers with parabolic law nonlinearity are constructed (see Table 2). The amplitude and width of the solitons are given by Eqs. (41) and (39), respectively, while the velocity of the solitons are given by Eq. (36) or (37) and finally the wave numbers are given by Eq. (33). The constraint conditions for analytical solutions to exist are given by Eqs. (40) and (42)–(44).

It needs to be noted that when the modulus \( m = 0 \) and \( m = 1 \), the Jacobian elliptic periodic traveling wave solutions become trigonometric periodic solutions (see Table 3), unbounded solutions (see Table 4), singular solutions (see Table 5), singular, bright and dark soliton solutions (see Table 6).
4. Conclusion

The Hirota equation, describing the propagation of optical solitons through birefringent fibers with Hamiltonian perturbations and parabolic law nonlinearity, is studied analytically by employing the Riccati equation expansion method and Jacobian elliptic equation expansion method. We report the Jacobian elliptic periodic traveling wave solutions, periodic solutions, unbounded solutions, singular solutions, singular, bright and dark soliton solutions. We obtain the constraint conditions for these solutions to exist.

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References


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